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Acoustic Fluid Flow through Holes and Permeability of Perforated Walls

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We first study the unsteady incompressible fluid flow through a hole in a wall in the two- and three-dimensional cases. In the first case, a convolution relation is obtained between the fluid flux through the hole and the difference of pressure between the far regions on the two sides of the wall. In the two-dimensional case, the pressure increases logarithmically with distance from the wall. In a second part, we study acoustic flow in a domain containing a wall with many small holes. The distance between two contiguous holes is of order η and the size of each hole, ε (η and ε are two small parameters). In the three-dimensional case the critical behaviour appears for $\varepsilon = \eta^2$: it is described by a convolution relation between the flow through the wall and the jump of pressure. In the two-dimensional case, the critical behaviour appears if $\eta \log \varepsilon$ tends to a constant; there is a differential relation between the flow through the wall and the jump of pressure.

1. INTRODUCTION

This paper is devoted to two related questions. The first (Sections 2 and 3) is the unsteady incompressible viscous flow through a hole in a wall. The second (Sections 4 and 5) is the application of asymptotic methods to the study of the acoustic flow through a wall containing many small holes; it appears that the "local" associated problems are those of the preceding sections.

Viscous flow through a hole in a wall was formely considered by Heywood [1], mostly in the steady-state case. He showed important differences with the classical problem of fluid flow around a bounded body [2, 3], as well as differences between the three- and two-dimensional cases. Our treatment of the three-dimensional case (Section 2) is a little different from Heywood's and is suitable to consider the unsteady problem, in particular the relation (of the time-convolution kind) between the fluid flux through the wall and the jump of pressure (between the far regions on the two sides of the wall).

We then consider (Section 3) an analogous problem in the two-

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dimensional case. The properties of the appropriate functional spaces in this case are very different from that of the three-dimensional case; in particular, flow with non-zero flux through the hole does not belong to an L^2-space. We consider this flow as a non-homogeneous boundary-value problem, where the non-homogeneous part is taken from the explicit solution for a point-source in a wall studied by Tuck [4]. We study this problem only in the case of a pulsating flow (it depends on time by the factor $e^{-i\omega t}$). Our treatment is not completely satisfactory and it deserves deeper mathematical study. It appears that the corresponding pressure grows logarithmically with distance from the hole; The first term of the pressure is independent of viscosity. Other problems about Navier–Stokes equations in regions with unbounded boundary may be seen in [17].

In the second part (Sections 4 and 5) we consider acoustic flow (i.e., linearized unsteady motion for a compressible, slightly viscous fluid) in a region Ω^0 containing a wall Σ perforated by holes of size $O(\varepsilon)$ spaced by distances $O(\eta)$, where ε and η are small coefficients tending to zero and such that $\varepsilon/\eta \to 0$. This problem belongs to the large class of problems in a domain with a granular boundary (see [5, 6]). We study our problem by using the method of matched asymptotic expansions (see [7, 8] for other problems involving flow through small holes and [9] for other associated problems in physics). If the holes are asymptotically very small (resp. large), the perforated wall behaves as an impermeable (resp. inexistent) wall. The critical case (i.e., the nontrivial limit behaviour) appears when

$$\begin{aligned} \varepsilon/\eta^2 &= O(1) \qquad \text{(three-dimensional case)}, \\ \eta \log \varepsilon &= O(1) \qquad \text{(two-dimensional case)}. \end{aligned} \tag{1.1}$$

In the critical case, there is a relation between the velocity flow through the wall and the pressure jump across it (i.e., an "impedance" condition). This relation is convolution-like (resp. differential) in the three-dimensional (resp. two-dimensional) case. We then study the existence and uniqueness of the solution of the limit problem (Section 4.5). The convolution boundary condition is taken into account by considering simultaneously the limit flow in Ω^0 and the local flow in the vicinity of the holes (i.e., we consider simultaneously the outer and inner limits); this method was first introduced by Lions [10, 11] for the study of homogenization problems involving convolution terms (see also [12, Section 6.8]).

We now give some generalities about notation:

Points in $\mathbb{R}^N$ ($N = 2$ or 3) are denoted by x, y, z, with

$$x = (x_1, \dots, x_N).$$

Other vectors are written in boldface:

$$\mathbf{V} = (V_1, ..., V).$$

Sobolev spaces are denoted in the standard way:

$$L^2(\Omega), \quad H^1, \quad H_0^1, ..., \qquad \text{and} \qquad \mathbf{L}^2(Q) = (L^2(Q))^N.$$

If $\varphi = \varphi(x, y)$, $\mathbf{grad}_x \varphi$ means the gradient when y is taken as a parameter. $|f|$ denotes the jump of the function f through a surface of discontinuity.

2. Viscous Incompressible Flow through a Hole: Three-Dimensional Case

2.1. *Setting of the Problem*

In the $\mathbb{R}^3$ space of the variables (z_1, z_2, z_3) we consider the plane $z_3 = 0$ as a "wall" where there is a "hole" S which is a bounded, open, and connected domain (of the z_1, z_2-plane) with smooth boundary ∂S. The origin $z = 0$ is supposed to belong to S.

The domain

$$\mathbb{R}_S^3 = \{z \in R^3;\ \text{either } z_3 \neq 0 \text{ or } (z_1, z_2) \in S\}, \tag{2.1}$$

formed by two semi-spaces binded by the hole, is filled by an incompressible (either viscous or inviscid) fluid in unsteady motion. If $\mathbf{V}(z, t)$, $P(z, t)$ denote the velocity and pressure, we consider the following initial-boundary value problem (which we state only in a formal form for the time being).

Find $\mathbf{V}(z, t)$, $P(z, t)$ defined on $\mathbb{R}_S^3 \times]0, \infty[$ satisfying

$$\rho \frac{\partial \mathbf{V}}{\partial t} = -\mathbf{grad}\, P + \lambda \Delta \mathbf{V}, \tag{2.2}$$

$$\operatorname{div} \mathbf{V} = 0, \tag{2.3}$$

$$\mathbf{V} \to 0 \quad \text{if} \quad |z| \to \infty, \tag{2.4}$$

$$P \to p^{0\pm}(t) \quad \text{if} \quad |z| \to \infty \quad \text{with} \quad z_3 \to \pm\infty, \tag{2.5}$$

$$\mathbf{V} = 0 \quad \text{on} \quad \partial R_S^3 \quad \text{if} \quad \lambda > 0, \tag{2.6a}$$

$$V_3 = 0 \quad \text{on} \quad \partial R_S^3 \quad \text{if} \quad \lambda = 0, \tag{2.6b}$$

$$\mathbf{V}(z, 0) = \text{given function}. \tag{2.7}$$

Here, λ is the viscosity coefficient, which may be taken either > 0 (viscous case) or $= 0$ (inviscid case). The vector $\mathbf{V}$ is subject to the no-slip condition on the wall (2.6a) in the viscous case and to the slip condition (2.6b) in the impermeable case. The pressure is subject to tend to two different, given functions of t, $p^{0\pm}(t)$, as $|z|$ tends to infinity in the regions $z_3 \gtrless 0$, as indicated by (2.5). The velocity field tends to zero at infinity; consequently the $\mathbf{V}$, P fields tend to two rest states with different pressure far from the wall; note that this is consistent with Eq. (2.2).

We shall see in the following sections that, after an appropriate mathematical definition of this formal problem, the solution $\mathbf{V}$, P exists and is unique. Moreover, the flux of $\mathbf{V}$ through the hole S, defined by

$$\Phi(\mathbf{V}, t) = \int_S V_3 \, dz_1 \, dz_2 \tag{2.8}$$

is a well-determined functional of the convolution type of the jump of pressure defined by

$$[p^0](t) = p^{0+}(t) - p^{0-}(t). \tag{2.9}$$

Remark 2.1. The hypothesis that S is connected is in fact unnecessary. We may also consider a finite number of hole in the wall; the treatment is exactly the same.

Remark 2.2. With slight, obvious modifications, we may also consider the case of a wall with non-zero thickness, constant for sufficiently large (z_1, z_2).

Remark 2.3. The non-linear terms of the classical Navier–Stokes equations were neglected in (2.2). This is consistent with our assumption of acoustic (linear flow), as we shall see in Section 4. Nevertheless, the non-linear case may also be considered and it leads to the classical non-uniqueness situations for large Reynolds number (see Ladyzhenskaya [2], Tenam [3], Heywood [1]).

2.2. *Function Spaces*

$\mathbf{L}^2(\mathbb{R}^3_S)$ are the space of the vectors $\mathbf{V}$ with components in $L^2(\mathbb{R}^3_S)$ (equivalent to $L^2(R^3)$). We consider the following subspaces of it:

$$H = \{\mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3_S);\ \operatorname{div} \mathbf{U} = 0,\ U_3|_{\partial\mathbb{R}^3_S} = 0\}, \tag{2.10}$$

$$V = \{\mathbf{U} \in \mathbf{H}^1_0(\mathbb{R}^3_S);\ \operatorname{div} \mathbf{U} = 0\}. \tag{2.11}$$

They are closed subspaces of $\mathbf{L}^2$ and $\mathbf{H}^1_0$, respectively, and then Hilbert spaces for the corresponding scalar products.

Remark 2.4. Note that the domain $\mathbb{R}^3_S$ does not have a smooth boundary, and it is not sure that V is a dense subset of H. This density property certainly holds in the case of Remark 2.2. If the property does not hold, H must be replaced by the closure of V in $\mathbf{L}^2$, which is contained in H. This question is immaterial in the sequel, and we shall consider that *V is dense in H.*

LEMMA 2.1. *There exists a vector $\mathbf{U}$ contained in H and V with non-zero flux (defined as in (2.8)) through the hole S.*

Proof. We consider polar coordinates r, θ, where θ are "angular coordinates" on the unit sphere S_0^2. We construct the vector $\mathbf{U}$ in the following way. Let $\varphi(\theta)$ be a smooth function such that

$$\int_{S^2} \varphi(\theta)\, dS_2 = 0, \tag{2.12}$$

$$\int_{S^2 \cap \{z; z_3 > 0\}} \varphi(\theta)\, dS_2 = -\int_{S^2 \cap \{z; z_3 < 0\}} \varphi(\theta)\, dS_2 = 1 \tag{2.13}$$

and φ is zero in the vicinity of $z_3 = 0$. Then, we define $\mathbf{U}$ in the region $r > 1$ by

$$\mathbf{U} = (\varphi(\theta)/r^2)\, \mathbf{e}_r, \tag{2.14}$$

where $\mathbf{e}_r$ is the unit vector in the direction r. It is clear that (2.14) may be made continuous in the region $r < 1$ by a divergence-free smooth function with zero values on $\partial \mathbb{R}_S^3$ (for (2.12) is the compatibility condition; see, for instance, Temam [3, Chap. 1, Theorem 2.4]). The vector $\mathbf{U}$ satisfies the conditions of the lemma. In particular, it is zero on $\partial \mathbb{R}_S^3$ and divergence free; then its flux through S is equal to its flux through the semi-sphere $r = R$, $z_3 > 0$ with sufficiently large R, whose value is 1 by virtue of (2.13), (2.14). ∎

Remark 2.5. The analog of the preceding lemma does not hold in the two-dimensional case, where we should take r instead of r^2 in the denominator of (2.14); the corresponding $\mathbf{U}$ does not belong to $\mathbf{L}^2$. This is the principal difference between the two- and three-dimensional cases.

LEMMA 2.2. *Let q be a smooth function defined on $\mathbb{R}_S^3$ such that for sufficiently large $|z|$ it takes the value* 1 (*resp.* 0) *for* $z_3 > 0$ (*resp.* < 0). *Then,*

$$\int_{\mathbb{R}_S^3} \frac{\partial q}{\partial z_i} V_i\, dz \tag{2.15}$$

is a bounded non-zero functional on H. Consequently, this functional is *expressed by the scalar product by a well-determined non-zero element* **F** *of H. Moreover, the value of this functional is equal to the flux of* **V** *through the hole, i.e.,*

$$(\mathbf{F}, \mathbf{V})_H = \int_{\mathbb{R}^3_S} \frac{\partial q}{\partial z_i} V_i \, dz = \int_S V_3 \, dz_1 \, dz_2 = \Phi(\mathbf{V}). \tag{2.16}$$

Proof. The support of $\mathbf{grad}\, q$ is bounded, and consequently, (2.15) is a bounded functional on H. Moreover, the vector $\mathbf{V}$ of H is divergence-free, and by integrating on a domain D containing the support of $\mathbf{grad}\, q$, we have

$$\int_D \frac{\partial q}{\partial z_i} V_i \, dz = \int_{\partial D} q V_i n_i \, dS, \tag{2.17}$$

where $\mathbf{n}$ is the outer unit normal to ∂D. But q takes the value 1 (resp. 0) on the region $z_3 > 0$ (resp. < 0) of ∂D, and consequently the value of the integral in (2.17) is the flux of $\mathbf{V}$ through the portion $z_3 > 0$ of ∂D; by again using the divergence-free condition this flux is equal to the flux through S, and (2.16) follows. Finally, the functional is not zero as we see by acting it on the vector $\mathbf{U}$ of Lemma 2.1. ∎

According to the standard theory of Navier–Stokes equations, the pressure term in (2.2) will be eliminated by orthogonality. We now consider the orthogonal complement $H^\perp$ of H in $\mathbf{L}^2(\mathbb{R}^3_S)$ and we prove that, in some generalized sense it consists of the gradients of the functions which tend to a constant at infinity (the same constant in the two regions z_3 more or less than zero!).

LEMMA 2.3. *Let $H^\perp$ be the orthogonal complement of H in $\mathbf{L}^2(\mathbb{R}^3_S)$. If $\mathbf{V} \in H^\perp$, there exists a function Q belonging locally to H^1, defined up to an additive constant, and such that there exists a constant c such that*

$$\int_{\{z, |z_3| > 1\}} (Q - c)^6 \, dz < \infty, \tag{2.18}$$

$$\int_{\{z, |z_3| > 1\}} \frac{(Q - c)^2}{|z|^2} \, dz < \infty, \tag{2.19}$$

and $\mathbf{V} = \mathbf{grad}\, Q$.

Proof. If $\mathbf{V} \in H^\perp$, by taking the restriction to a bounded domain D, we see that

$$\int_D V_i w_i \, dz = 0$$

for any $\mathbf{w}$ divergence free smooth vector with compact support in D, and according to the standard theory, $\mathbf{V} = \mathbf{grad}\, Q$. Now, we consider the function $\tilde{Q}$ which coincides with Q for $z_3 > 1$ and is prolongated by reflexion for $z_3 < 1$.

$$\begin{aligned}\tilde{Q}(z_1, z_2, z_3) &= Q(z_1, z_2, z_3) \qquad \text{if} \quad z_3 \geqslant 1 \\ &= Q(z_1, z_2, 2 - z_3) \qquad \text{if} \quad z_3 < 1.\end{aligned} \tag{2.20}$$

It is clear that $\mathbf{grad}\, \tilde{Q}$ belongs to $\mathbf{L}^2(\mathbb{R}^3)$, according to the classical inequalities of Sobolev and Finn (see, for instance, Ladyzhenskaya [2], and Finn [13] or Heywood [1]) there exists a constant c such that

$$\int_{\mathbb{R}^3} (\tilde{Q} - c)^6 \, dz < \infty; \qquad \int_{\mathbb{R}^3} \frac{(\tilde{Q} - c)^2}{|z|^2} \, dz < \infty. \tag{2.21}$$

This means that Q "tends" to c for large $|z|$ in the region $z_3 > 1$. An analogous proof shows that Q "tends" to another constant c' in the region $z_3 < 1$. Finally, the two constants c and c' are the same; for, if $c \neq c'$ by reasoning as in the proof of Lemma 2.2 we see that

$$(\mathbf{U}, \mathbf{grad}\, Q)_{L^2} = (c - c')\, \Phi(\mathbf{U}),$$

which is $\neq 0$ by taking the vector $\mathbf{U}$ of Lemma 2.1 and consequently $\mathbf{grad}\, Q \notin H^\perp$. Then lemma is then proved. ∎

2.3. *Resolution of Problem* (2.2)–(2.7) *and Consequences*

We now search for an abstract functional formulation of problem (2.2)–(2.7). We search for $\mathbf{V}(t)$ under the form of a function of t with values in H (and in V in the viscous case $\lambda > 0$). Conditions (2.3), (2.4), and (2.6) are then automatically fulfilled.

Moreover, by using the function q of Lemma 2.2 and definition (2.9) of the jump of p^0, we define the new unknown Q instead of P by

$$Q = P - |p^0|\, q \tag{2.22}$$

and condition (2.5) and Eq. (2.2), respectively, become

$$Q \to p^{0-} \qquad \text{if} \quad |z| \to \infty \tag{2.23}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} - \lambda\, \Delta \mathbf{V} = -\mathbf{grad}\, Q - |p^0|\, \mathbf{grad}\, q. \tag{2.24}$$

We multiply (2.24) by a test function $\mathbf{w} \in V$, and integrating by parts we have

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}, \mathbf{w}\right)_H + \lambda a(\mathbf{V}, \mathbf{w}) = -[p^0](\mathbf{F}, \mathbf{w})$$
$$a(\mathbf{V}, \mathbf{w}) = \int_{\mathbb{R}^3_S} \frac{\partial V_i}{\partial z_k} \frac{\partial w_i}{\partial z_k}\, dz, \tag{2.25}$$

where $\mathbf{F}$ is the element of H defined by (2.16). Conversely if $\mathbf{V}$ satisfies (2.25) for any $\mathbf{w} \in V$ and $\Delta V \in L^2$, we see by Lemma 2.3, that Q satisfying (2.22) exists and is well determinated (note that function Q in lemma 2.3 is determinated up to an additive constant, which may be chosen for each t in order to satisfy (2.22)).

We then are in the standard situation of parabolic equation. If A is the self-adjoint operator of H associated with the form $a(\mathbf{V}, \mathbf{w})$, (2.25) amounts to

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \lambda A \mathbf{V} = -[p^0](t)\, F \tag{2.26}$$

and we solve (2.26), (2.7) by standard semigroup theory:

$$\mathbf{V}(t) = e^{-(\lambda/\rho)At}\mathbf{V}(0) - \frac{1}{\rho}\int_0^t e^{-(\lambda/\rho)A(t-s)}[p^0](s)\, F\, ds. \tag{2.27}$$

We see that the "boundary conditions" $[p^0](s)$ for $s \in\,]0, t[$ and the initial values $\mathbf{V}(0)$ determine $\mathbf{V}(t)$ and in particular the flux of $\mathbf{V}$ through the hole is defined by (see (2.16))

$$\begin{aligned}\Phi(\mathbf{V}(t)) &= (\mathbf{F}, \mathbf{V}(t))_H \\ &= (e^{-(\lambda/\rho)At}\mathbf{V}(0), \mathbf{F})_H - \int_0^t g(t-s)[p^0](s)\, ds,\end{aligned} \tag{2.28}$$

where $g(\xi) = (e^{-(\lambda/\rho)A\xi}\mathbf{F}, \mathbf{F})_H/\rho$.

We have then proved

PROPOSITION 2.1. *Problem* (2.2)–(2.7) *has a unique solution given by* (2.27). *In particular the flux through S is a functional of $[p^0]$ given by* (2.28).

Remark 2.6. It is clear that (2.27), (2.28) hold in the inviscid as well as in the viscous case. In the inviscid case $\lambda = 0$, and $\mathbf{V}$ is merely the primitive function of $[p^0](t)\, F$ (see (2.26)).

3. Pulsating Incompressible Flow through a Hole: Two-Dimensional Case

3.1. *Setting of the Problem*

We consider here a problem analogous to that of the preceding sections but in the two-dimensional case. It appears that several features, in particular, the behaviour of pressure at infinite, are very different from those of the three-dimensional case. In particular, we shall take as a datum the flux through the hole (instead of the pressure jump, as in Section 2.1). The mathematical study is not completely satisfactory in this case, and a deeper treatment should be useful. We study the present problem by modifying the explicit solution for a point source in a wall obtained by Tuck [4] in the case of sinusoidal dependence on time (for the steady-state case, see Heywood [1, Theorem 12)]. Let S be a segment of the $0z_2$ axis, containing the origin. We consider the domain

$$\mathbb{R}^2_S = \{z \in \mathbb{R}^2;\ z_2 \neq 0 \text{ or } z_1 \in S\} \tag{3.1}$$

made of two semi-planes joined by the hole S (a wall with non-zero thickness may also be considered, as in Remark 2.2). If the velocity and pressure are $e^{-i\omega t}\mathbf{V}$ and $e^{-i\omega t}P$, the equations and boundary conditions of the problem are

$$-i\omega\rho\mathbf{V} = -\mathbf{grad}\, P + \lambda\, \Delta\mathbf{V} \tag{3.2}$$

$$\operatorname{div} \mathbf{V} = 0, \tag{3.3}$$

$$V \to 0 \qquad \text{if} \quad |z| \to \infty, \tag{3.4}$$

$$\Phi(\mathbf{V}) \equiv \int_S V_2\, dz_1 = 1 \tag{3.5}$$

$$\begin{cases} \mathbf{V} = 0 & \text{on } \partial\mathbb{R}^2_S \quad \text{if} \quad \lambda > 0, \\ V_2 = 0 & \text{on } \partial\mathbb{R}^2_S \quad \text{if} \quad \lambda = 0, \end{cases} \tag{3.6}$$

where the flux through the hole is equal to one (3.5). Because of the linearity, any other flux may also be imposed.

Our aim is to prove the existence and uniqueness of $\mathbf{V}$, P and the behaviour of P at infinity.

3.2. *Reduction to a Variational Problem*

We define the spaces H and V exactly as in (2.10), (2.11) but with the domain $\mathbb{R}^2_S$ instead of $\mathbb{R}^3_S$. We then have (compare with Lemma 2.1):

LEMMA 3.1. *If* $\mathbf{U} \in H$ *(two-dimensional case), the flux of* $\mathbf{U}$ *through the hole,*

$$\Phi(\mathbf{U}) \equiv \int_S U_2 \, dz_1 \tag{3.7}$$

is zero.

Proof. Because $\mathbf{U} \in H$, it is divergence free and its normal component on $\partial \mathbb{R}^2_S$ is zero. Consequently, its flux (3.7) is the same as the flux through a semi-circumference $|z| = R$, $z_2 > 0$ for sufficiently large R. We shall prove the lemma by contradiction: If the flux is $K \neq 0$ by taking polar coordinates r, θ, we have

$$|K| = \int_0^\pi |U_r| \, r \, d\theta \leqslant \left(\int_0^\pi |U_r|^2 \, r \, d\theta \right)^{1/2} \left(\int_0^\pi r \, d\theta \right)^{1/2}$$

$$\Rightarrow \frac{|K|^2}{\pi r} \leqslant \int_0^\pi |U_r|^2 \, r \, d\theta$$

$$\Rightarrow \int_c^\infty dr \int_0^\pi |U_r|^2 \, d\theta \geqslant \frac{|K|^2}{\pi} \int_c^\infty \frac{dr}{r}; \tag{3.8}$$

but the right-hand side of (3.8) diverges and consequently the left-hand side does, and we have a contradiction with $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}^2_S)$. ∎

Now we construct auxiliary "velocity vector" $\mathbf{a}$ and "pressure" q satisfying (3.3)–(3.6) and Eq. (3.2) *in a neighborhood of infinity*. To this end, we consider vector $\tilde{\mathbf{a}}$ and the corresponding pressure $\tilde{q}$ given by Tuck [4] for a source with unit intensity in a wall. The corresponding pressure is defined up to an additive constant; we fix it in an arbitrary way. We construct $\mathbf{a}$ equal to $\tilde{\mathbf{a}}$ in the region $z_2 > 0$ for sufficiently large $|z|$. In the same way, we define $\mathbf{a}$ in the region $z_2 < 0$ for sufficiently large $|z|$ equal to the solution for a unit sink in a wall (obtained from the source by changing signs). We then continuate $\mathbf{a}$ (which is only defined for sufficiently large $|z|$ for the time being) to all $\mathbb{R}^2_S$ satisfying (3.3) and (3.6). This is possible because the compatibility condition concerning the total flux is obviously satisfied. We then define q in an analogous way, equal to $\tilde{q}$ for $z_2 > 0$, and sufficiently large $|z|$, equal to the corresponding pressure for a sink for $z_2 < 0$ and we prolongate it in an arbitrary way to $\mathbb{R}^2_S$.

We now consider the new unknowns $\mathbf{U}$, Q defined by

$$\mathbf{U} = \mathbf{V} - \mathbf{a}; \qquad Q = P - q \tag{3.9}$$

instead of $\mathbf{V}$, P. The system (3.2)–(3.6) becomes

$$-i\omega\rho\mathbf{U} = -\mathbf{grad}\, Q + \lambda\, \Delta\mathbf{U} + \mathbf{f}, \tag{3.10}$$

$$\operatorname{div} \mathbf{U} = 0, \tag{3.11}$$

$$\mathbf{U} \to 0 \quad \text{as} \quad |z| \to \infty, \tag{3.12}$$

$$\Phi(\mathbf{U}) = \int_S U_2\, dz_1 = 0, \tag{3.13}$$

$$\mathbf{U} = 0 \quad \text{on} \quad \partial\mathbb{R}_S^2 \quad \text{if} \quad \lambda > 0, \tag{3.14a}$$

$$U_2 = 0 \quad \text{on} \quad \partial\mathbb{R}_S^2 \quad \text{if} \quad \lambda = 0, \tag{3.14b}$$

where $\mathbf{f}$ is a smooth function defined on $\mathbb{R}_S^2$ and *equal to zero in a neighborhood of infinity*.

To solve (3.10)–(3.14), we search for $\mathbf{U}$ as an element of V (or of H in the inviscid case $\lambda = 0$) and (3.11)–(3.14) are satisfied. Equation (3.10) is obviously equivalent to

$$-i\omega\rho(\mathbf{U}, \mathbf{w})_H + \lambda a(\mathbf{U}, \mathbf{w}) = \int_{\mathbb{R}_S^2} f_i w_i\, dz \qquad \forall\, \mathbf{w} \in V \tag{3.15}$$

where the form $a(\mathbf{U}, \mathbf{w})$ is defined as in (2.25). It is clear that in the inviscid case $\lambda = 0$, (3.15) may be written for any $\mathbf{w} \in H$. Problem (3.2)–(3.6) becomes:

Abstract Problem. Find $\mathbf{U} \in V$ (or H in the inviscid case $\lambda = 0$) such that (3.15) is satisfied for any $\mathbf{w} \in V$ (or H in the inviscid case). Then, $\mathbf{V}$ and P are defined by (3.9).

3.3. *Resolution of the Problem and Consequences*

The right-hand side of (3.15) is a linear bounded functional on either V or H (because $\mathbf{f}$ is zero in a neighbourhood of infinity), and the left-hand side is obviously a sesquilinear, coercive form on V (resp. H) for $\lambda > 0$ (resp. $\lambda = 0$). Consequently, the Lax–Milgram theorem gives:

PROPOSITION 3.1. *The vector* $\mathbf{U}$ (*and consequently* $\mathbf{V}$, *defined by* (3.9)) *exists and is unique. The corresponding pressure* Q (*and consequently* P *defined by* (3.9)) *exists and is unique up to an additive constant.*

To study the behaviour of the pressure P for large $|z|$, we see, by taking the divergence of (3.2) and taking into account (3.3), that P is a harmonic function in $\mathbb{R}_S^2$. In the same way, we see from (3.10) that Q is a harmonic function in $\mathbb{R}_S^2$ for large $|z|$.

Let us consider Q for large $|z|$ in the region $z_2 > 0$. *We assume that Q admits in this region the classical expansion*

$$Q = \gamma_0 + \alpha_0 \log r + \beta_0 \theta$$
$$+ \sum_1^{\infty} [(\alpha_n r^{-n} + \tilde{\alpha}_n r^n) \cos n\theta + (\beta_n r^{-n} + \tilde{\beta}_n r^n) \sin r\theta] \qquad (3.16)$$

for $z_2 > 0$, sufficiently large $|z|$.

Remark 3.1. Expansion (3.16) is fully rigorous in the inviscid case $\lambda = 0$, since, from (3.10) and (3.14b) we see that Q satisfies a boundary condition of the Neumann type $\partial Q/\partial n = 0$. Consequently, Q may be continuated to the region $z_2 < 0$ by $Q(z_1, z_2) = Q(z_1, -z_2)$ and the continuated function is harmonic in a neighbourhood of infinity. According to the classical theory, expansion (3.16) holds. On the other hand, in the viscous case $\lambda > 0$, we do not know the boundary conditions satisfied by Q, and it is not clear that Q admits a continuation. In this case, (3.16) is merely a formal expansion in separable variables. It seems that, in this case, n may take non-integer values (as Q may not be a uniform function) but in any case the behaviour with respect to r will be that of (3.16), with real values of n. Under this condition, the following considerations hold.

This enables us to give the behaviour of P as $|z| \to \infty$. We have:

PROPOSITION 3.2. *Under assumption* (3.16) (see Remark 3.1), *there exists a constant Γ (which depends on ρ, ω, λ and the hole S) such that the pressure P, in the solution of* (3.2)–(3.6), *converges as $|z| \to \infty$ to*

$$P \to \pm \left(\frac{i\omega\rho}{\pi} \log|z| + \Gamma\right) + c, \qquad (3.17)$$

where the sign + (resp. −) works for $z_2 > 0$ (resp. $z_2 < 0$) and constant c is unessential (as P is determined up to an additive constant).

Remark 3.2. The term log in (3.17) corresponds to an inviscid source (resp. sink) of intensity 1 (see (3.5)) in the region $z_2 > 0$ (resp. $z_2 < 0$).

Proof of Proposition 3.2. We first study the behaviour of Q in the region $z_2 > 1$. Because $\mathbf{U} \in \mathbf{L}^2$ $(z_2 > 1)$ a standard reasoning (see Heywood [1, Lemmas 9, 3)] shows that **grad** Q (as well as the second derivatives of $\mathbf{U}$) belongs to L^2 $(z_2 > 1)$.

It follows that the singular terms of (3.16) vanish and consequently, Q tends to a constant as $|z| \to \infty$. The same reasoning in $z_2 < -1$ shows that Q tends to another constant at infinity in this region. The difference of these constants is of course well determinated by the data of the problem.

Let us now consider function q. Its behaviour at infinity is the same as that of the "pressure" $\tilde{q}$ for the source solution of Tuck [4] (see the construction of q). If we consider the stream function ψ defined by

$$\tilde{U}_1 = \partial\psi/\partial z_2; \qquad \tilde{U}_2 = -\partial\psi/\partial z_1,$$

the pressure $\tilde{q}$ is the conjugate harmonic function of $-i\omega\psi - (\lambda/\rho)\Delta\psi$. Its asymptotic behaviour immediatly follows from Tuck ([4]: (3.5), (4.1), (4.3), where $\omega = -\Delta\psi$). It follows that $\tilde{q}$ behaves at infinity (for $z_2 > 0$) as

$$\frac{i\omega\rho}{\pi}\log|z| + (\text{vanishing terms as } |z| \to \infty),$$

where an additive constant has been fixed (note that this is consistent with the construction of q and Q). The behaviour (3.17) immediately follows.

4. Acoustic Flow through a Perforated Wall: Three-Dimensional Case

4.1. *Setting of the Problem*

We consider a domain Ω^ε formed by two domains separated by a wall perforated by many small holes, defined in the following way.

Let Ω^0 be a bounded domain, with smooth boundary, of the space R^3 (coordinates x_1, x_2, x_3). The origin is contained in Ω^0.

Let ε, η be two small parameters. We shall consider later that ε, η are functions of each other; for the time being, we only assume

$$\varepsilon, \eta \to 0; \qquad \varepsilon/\eta \to 0. \tag{4.1}$$

In the plane $x_3 = 0$ (the "wall") we consider a periodic array of points (periods ηa and ηb)

$$x_1 = \eta a n; \qquad x_2 = \eta b m \qquad (m, n \text{ integers}), \tag{4.2}$$

where a and b are two given numbers.

On the other hand, we consider in the plane of the variables z_1, z_2 a "hole" S (bounded domain containing the origin, defined as in Section 2.1). Then, in the plane x_1, x_2, we consider a system of holes εS (homothetic of S) centered at the points (4.2). Roughly speaking, we have holes of size ε periodically located at distances of order η (note that, by virtue of (4.1) the distance between two adjacent holes is large with respect to the size of the hole).

The domain Ω^ε is defined as the domain obtained by removing from Ω^0 the perforated wall (i.e., is formed by the points of Ω^0 with $x_3 \neq 0$ and the points with $x_3 = 0$ contained in the holes). This domain depends on ε (and also on η, but as we already said, η will be considered a function of ε).

We search for the functions $\mathbf{v}^\varepsilon(x, t)$, $p^\varepsilon(x, t)$ defined for $x \in \Omega^\varepsilon$ and $t \in]0, \infty[$ satisfying the initial boundary value problem

$$\rho \frac{\partial \mathbf{v}^\varepsilon}{\partial t} = -\mathbf{grad}\, p^\varepsilon + \varepsilon^2 \lambda \Delta \mathbf{v}^\varepsilon + \varepsilon^2 \mu\, \mathbf{grad} \operatorname{div} \mathbf{v}^\varepsilon + \mathbf{f}, \tag{4.3}$$

$$\frac{\partial p^\varepsilon}{\partial t} + \rho c^2 \operatorname{div} \mathbf{v}^\varepsilon = 0, \tag{4.4}$$

$$\mathbf{v}^\varepsilon = 0 \qquad \text{on} \quad \partial\Omega^\varepsilon, \tag{4.5}$$

$$\mathbf{v}^\varepsilon = p^\varepsilon = 0 \qquad \text{for} \quad t = 0, \tag{4.6}$$

where ρ, λ, μ, c^2 are given positive constants and $\mathbf{f}(x, t)$ is a given function of x, t. We assume that it is zero for x in a neighbourhood of the wall $x_3 = 0$, and any t.

Remark 4.1. Problem (4.3)–(4.6) is that of the acoustic (linearized) vibrations of a barotropic gas with small viscosity ($\varepsilon^2\lambda$, $\varepsilon^2\mu$ are the viscosity coefficients). In this case, p^ε and $\mathbf{v}^\varepsilon$ are, respectively, perturbations of pressure and velocity. The boundary condition (4.5) is the classical no-slip condition of viscous fluids.

Remark 4.2. We shall see later that the critical case is $\varepsilon = \eta^2$. Other cases are considered in Section 6. The viscosity coefficients $\varepsilon^2\lambda$ and $\varepsilon^2\mu$ then lead to the viscous problem in the vicinity of the holes. Other orders may also be considered. On the other hand, we may consider $\lambda = \mu = 0$. In this case, (4.5) must be replaced by

$$\mathbf{v}^\varepsilon \cdot \mathbf{n} = 0, \tag{4.7}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega^\varepsilon$.

Remark 4.3. The hypotheses on Ω^0 and $\mathbf{f}$ are unessential. Unbounded domains may also be considered. Instead of $\mathbf{f}$ we also may consider non-zero initial or boundary conditions.

Remark 4.4. We assume the wall to be the plane $x_3 = 0$. More generally a curved wall can be considered. Indeed, in the ε-dilatation any small portion of the surface is equivalent to its tangent plane.

4.2. *Outer Expansion*

As $\varepsilon \to 0$ a boundary layer appears in the vicinity of $\partial\Omega^\varepsilon$; out of this region we consider an expansion (outer expansion) of the form

$$p^\varepsilon(x, t) = p^0(x, t) + \varepsilon p^1(x, t) + \cdots, \tag{4.8}$$

$$\mathbf{v}^\varepsilon(x, t) = \mathbf{v}^0(x, t) + \varepsilon \mathbf{v}^1(x, t) + \cdots, \tag{4.9}$$

and (4.3), (4.4), (4.6) give, at order ε^0,

$$\rho \frac{\partial \mathbf{v}^0}{\partial t} = -\mathbf{grad}\ p^0 + \mathbf{f}, \tag{4.10}$$

$$\frac{\partial p^0}{\partial t} + \rho c^2 \operatorname{div} \mathbf{v}^0 = 0, \tag{4.11}$$

$$\mathbf{v}^0(x, 0) = p^0(x, 0) = 0. \tag{4.12}$$

The appropriate boundary conditions will be obtained by matching with the boundary layer solutions. The boundary of Ω^ε consists of the standard wall $\partial\Omega^0$ enclosing the domain Ω^0 and of the region $x_3 = 0$ formed by the perforated wall. In the first, a classical oscillating boundary layer appears (see, for instance, Landau and Lifschitz [14, Section 24] and we have

$$\mathbf{v}^0 \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega^0. \tag{4.13}$$

Moreover, we assume that on the perforated wall,

$$[p^0] \neq 0; \qquad v_3^{0+} = v_3^{0-} \neq 0 \qquad (\text{on } x_3 = 0), \tag{4.14}$$

where

$$[p^0] = p^{0+}(x_1, x_2, t) - p^{0-}(x_1, x_2, t),$$

$$p^{0\pm}(x_1, x_2, t) = \lim_{x_3 \to 0\pm} p^0(x, t),$$

$$\mathbf{v}^{0\pm}(x_1, x_2, t) = \lim_{x_3 \to 0\pm} \mathbf{v}^0(x, t).$$

The meaning of (4.14) is the following. If the holes are (asymptotically) very small, the wall behaves in the limit as an impermeable wall, with a boundary condition of type (4.13), i.e., $v_3^{0\pm} = 0$ (where (4.10) has been taken into account). On the other hand, if the holes are (asymptotically) large the effect of the wall disappears in the limit, and the appropriate boundary conditions are

$$[p^0] = 0; \qquad v_3^{0+} = v_3^{0-} \neq 0. \tag{4.15}$$

Consequently, (4.14) are "intermediate" boundary conditions. This is the meaning of "critical case" in Remark 4.2. Moreover, the equality $v_3^{0+} = v_3^{0-}$ in (4.14) is a consequency of the fact that in the boundary layers the gas behaves as an incompressible fluid. In addition, we shall see that there is a relation between the jump of p^0 and v_3^0 (associated with (2.28)).

4.3. *Expansion in the η-Layer*

In the vicinity of $x_3 = 0$ (with $x_3 > 0$; an analogous study holds for $x_3 < 0$) we consider the new variable

$$y = x/\eta \tag{4.16}$$

as in the classical homogenization method for periodic structures [12, 15]. We consider a two-scale expansion, where the functions depend on $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3)$;

$$p^\varepsilon = \hat{p}^0(x, y, t) + \eta \hat{p}^1(x, y, t) + \cdots,$$
$$\mathbf{v}^\varepsilon = \hat{\mathbf{v}}^0(x, y, t) + \eta \hat{\mathbf{v}}(x, y, t) + \cdots,$$

and we obtain from (4.3), (4.4) at the first order

$$\mathbf{grad}_y\, \hat{p}^0 = 0 \Rightarrow \hat{p}^0 = \hat{p}^0(x_1, x_2, t), \tag{4.17}$$

$$\rho \frac{\partial \hat{\mathbf{v}}^0}{\partial t} = -\mathbf{grad}_x\, \hat{p}^0 - \mathbf{grad}_y\, \hat{p}^1, \tag{4.18}$$

$$\operatorname*{div}_y \hat{\mathbf{v}}^0 = 0. \tag{4.19}$$

At this order, (4.18), (4.19) show that the flow is incompressible and inviscid. If we consider x as a parameter, (4.18) may be written

$$\rho \frac{\partial \hat{\mathbf{v}}^0}{\partial t} = -\mathbf{grad}_y(\hat{p}^1 + \mathbf{y} \cdot \mathbf{grad}_x\, \hat{p}^0). \tag{4.20}$$

This equation, with (4.19), shows that the flow in the η-layer is that of an incompressible fluid with sources at the points

$$y_1 = an; \qquad y_2 = bm, \tag{4.21}$$

n, m integers, of the wall $y_3 = 0$ and tending for $y_3 \to +\infty$ to a uniform $\mathbf{v}^{0+}$ flow. This flow (in the case $v_i^{0+} = 0$, $i = 1, 2$) is studied in Tuck [8, Section VI-B]; $\hat{p}_1$ *is, up to a factor* γ, *the function* φ *of Tuck,*

$$\varphi(y_1, y_2, y_3) = -\frac{1}{4\pi a}\,\Phi\left(\frac{y_1}{a}, \frac{(y_2^2+y_3^2)^{1/2}}{a}\right)$$
$$-\frac{1}{4\pi a}\left\{\sum_{j=1}^{\infty}\Phi\left(\frac{y_1}{a}, \frac{((y_2-jb)^2+y_3^2)^{1/2}}{a}\right)\right.$$
$$\left.+\Phi\left(\frac{y_1}{a}, \frac{((y_2+jb)^2+y_3^2)^{1/2}}{a}\right) - 2\Phi\left(0, \frac{jb}{a}\right)\right\}, \quad (4.22)$$

where

$$\Phi(Y, R) = \frac{1}{(Y^2+R^2)^{1/2}} + \sum_{k=1}^{\infty}\left\{\frac{1}{((k-Y)^2+R^2)^{1/2}}\right.$$
$$\left.+\frac{1}{((k+Y)^2+R^2)^{1/2}} - \frac{2}{k}\right\}$$

whose behaviour at infinity is

$$\varphi \to \frac{y_3 + \text{const}}{2ab} \qquad (\text{as } y_3 \to +\infty). \quad (4.23)$$

The matching of (4.20) with the outer expansion gives

$$\lim_{y_3\to+\infty} \rho\frac{\partial \hat{v}_i^0}{\partial t} = -\frac{\partial p^0}{\partial x_i}, \qquad i = 1, 2, \quad (4.24)$$

$$\lim_{y_3\to+\infty} \rho\frac{\partial \hat{v}_3^0}{\partial t} = -\frac{\gamma}{2ab}. \quad (4.25)$$

We finally have

$$\lim_{y_3\to+\infty} \hat{v}_3^0 = v_3^{0+}(x_1, x_2, t),$$
$$\hat{p}^0(x, t) = p^{0+}(x_1, x_2, t),$$
$$\hat{p}^1(x, y, t) = -2ab\rho\frac{\partial v_3^{0+}}{\partial t}(x_1, x_2, t)\,\varphi(y) + c(x, t),$$

and consequently, in te η-layer we have

$$p^\varepsilon = p^{0+}(x_1, x_2, t) - 2ab\rho\frac{\partial v_3^{0+}}{\partial t}(x_1, x_2, t)\,\eta\varphi + 0(\eta), \quad (4.26)$$

where $\theta(\eta)$ is for higher orders in η.

4.4. *Expansion in the Vicinity of the Holes* (ε-*Region*)

We take the origin at the center of a hole and we introduce the local (inner variable) $z = x/\varepsilon$. We search for expansions of the form

$$\begin{aligned} \mathbf{v}^\varepsilon &= \mathbf{v}(x_1, x_2, z, t) + \cdots, \\ p^\varepsilon &= P(x_1, x_2, z, t) + \cdots, \end{aligned} \qquad z = x/\varepsilon. \tag{4.27}$$

Note that the preceding expansion does not depend on the variable y of the η-layer. We shall see later that this is a natural consequence of the matching conditions.

Equation (4.4) at the first order gives

$$\operatorname*{div}_z \mathbf{v} = 0, \tag{4.28}$$

which shows that at the first order, the flow in the hole is incompressible (as well as in the η-layer (4.19)). Consequently, the flux of $\mathbf{v}^\varepsilon$ through each hole is the same as the flux through the η-layer (i.e., v_3^{0+}). We thus have

$$\varepsilon^2 \int_S v_3 \, dz_1 \, dz_2 = \eta^2 ab v_3^{0+} \tag{4.29}$$

and consequently $\mathbf{v}$ in (4.27) must be of order $(\eta/\varepsilon)^2$. We shall take

$$\varepsilon = \eta^2 \tag{4.30}$$

(note that a constant factor in (4.30) is unessential; it may be included in the constants a, b). Consequently we shall write (4.27) in the form

$$\begin{aligned} \mathbf{v}^\varepsilon &= (1/\varepsilon)\, \mathbf{V}(x_1, x_2, z, t) + \cdots, \\ p^\varepsilon &= P(x_1, x_2, z, t) + \cdots, \end{aligned} \tag{4.31}$$

and (4.3) at the first order

$$\rho \frac{\partial \mathbf{V}}{\partial t} = -\mathbf{grad}_z P + \lambda \Delta \mathbf{V}; \qquad \operatorname*{div}_z \mathbf{V} = 0. \tag{4.32}$$

We now match P in (4.31) with the expansion (4.26) of p^ε in the η-layer. According to (2.22) the behaviour of φ for small y is $\varphi \sim (4\pi |y|)^{-1}$. By writing it in the z-variables, (4.26) gives, in the matching region,

$$p^\varepsilon \sim p^{0\pm}(x_1, x_2, t) \mp \frac{\rho ab}{2\pi} \frac{\partial v_3^{0+}}{\partial t} \frac{1}{|z|} + O(\varepsilon^{-1/2})$$

and the matching with (4.31) gives

$$\lim_{|z|\to\infty} P(x_1, x_2, z, t) = p^{0\pm}(x_1, x_2, t) \tag{4.33}$$

and it appears that the problem in the ε-region (i.e., (4.32), (4.33)) is exactly the problem (2.2)–(2.7) of Section 2 (with, of course, $V(z, 0) = 0$, as a consequence of (4.6)). In particular, from (4.29), (4.30), (4.31), (4.33), and (2.28) *we have a convolution relation between* v_3^{0+} *and* $[p^0]$

$$\begin{aligned} abv_3^{0+}(x_1, x_2, t) &= \int_S V_3\, dz_1\, dz_2 \\ &= -\int_0^t g(t-s)[p^0(x_1, x_2, s)]\, ds. \end{aligned} \tag{4.34}$$

Remark 4.5. If ε/η^2 tends to zero the matching for p^ε gives $v_3^{0\pm} = 0$ and the wall is impermeable in the limit. Conversely if ε/η^2 tends to infinity we obtain $[p^0] = 0$; the wall is inexistent.

4.5. *The "Outer" Problem: Existence and Uniqueness*

The outer expansion was given in Section 4.2. The unknowns are $\mathbf{v}^0$ and p^0, functions of (x_1, x_2, x_3, t) which satisfy Eqs. (4.10, (4.11), and initial conditions (4.12). As for the boundary conditions, we have (4.13) on $\partial\Omega$ and (4.14) on the wall $x_3 = 0$, (which will be noted by Σ in the sequel), where v_3^0 and $[p^0]$ are related by (4.34). This system is of course integro-differential and, in some sense, of "hyperbolic type." Instead of studying the existence and uniqueness of the solution under this form, we shall study the system formed by $\mathbf{v}^0$, p^0, and $\mathbf{V}$ simultaneously.

In order to simplify the treatment we shall take

$$a = b = \rho = c^2 = \lambda = 1. \tag{4.35}$$

Moreover, we write

$$\begin{aligned} \mathbf{v}^0 &= \partial\mathbf{u}^0/\partial t; \qquad \mathbf{u}^0(0) = 0, \\ \mathbf{V} &= \partial\mathbf{U}/\partial t; \qquad \mathbf{U}(0) = 0 \end{aligned} \tag{4.36}$$

and (4.11), (4.10), (4.13) become

$$p^0 = -\operatorname{div} \mathbf{u}^0 \tag{4.37}$$

$$\partial^2\mathbf{u}^0/\partial t^2 = \mathbf{grad} \operatorname{div} \mathbf{u}^0 + \mathbf{f}, \tag{4.38}$$

$$\mathbf{u}^0 \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega^0. \tag{4.39}$$

On the other hand, (2.26) becomes

$$\partial \mathbf{V}/\partial t + A\mathbf{V} = [\operatorname{div} \mathbf{u}^0]\, \mathbf{F}. \tag{4.40}$$

Moreover, in order to study the existence and uniqueness of the solution of the problem in terms of $\mathbf{u}^0$, $\mathbf{U}$ we shall write (4.34) in an equivalent form as a relation between u^0 and U. Indeed

$$u_3^0 = \int_S U_3 \, dz_1 \, dz_2 = (\mathbf{F}, \mathbf{U})_H \qquad \text{on } \Sigma. \tag{4.41}$$

Of course V and H are the spaces defined in (2.10), (2.11) and $\mathbf{F}$ is the element of H defined in (2.16).

Remark 4.6. Relation (4.34) is in fact the time-derivative of (4.41); they are equivalent by virtue of the initial conditions in (4.36). The same remark holds for the equivalence of (4.39) and (4.13).

At present, the problem is to find the $\mathbf{u}^0(x, t)$, $\mathbf{U}(z, x, t)$ (the last defined for $z \in R_S^3$ and $x \in \Sigma$) satisfying Eqs. (4.38), (4.40) with the boundary conditions (4.39) on $\partial\Omega^0$ and the coupling condition (4.41) on Σ (and of course, the initial values: zero for $\mathbf{u}^0$, $\mathbf{U}$ and their first time-derivatives).

In order to study this problem, we define $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$, $\mathbf{u}^4$ by

$$\mathbf{u}^1 = \mathbf{u}^0; \qquad \mathbf{u}^2 = \mathbf{U}; \qquad \mathbf{u}^3 = \partial\mathbf{u}^0/\partial t; \qquad \mathbf{u}^4 = \partial\mathbf{U}/\partial t \tag{4.42}$$

and the system becomes

$$\begin{cases} d\mathbf{u}^1/dt = \mathbf{u}^3, \\ d\mathbf{u}^2/\partial t = \mathbf{u}^4, \\ \partial\mathbf{u}^3/\partial t = \mathbf{grad}\operatorname{div}\mathbf{u}^1 + \mathbf{f}, \\ \partial\mathbf{u}^4/\partial t = -A\mathbf{u}^4 + [\operatorname{div}\mathbf{u}^1]\,\mathbf{F} \end{cases} \tag{4.43}$$

with boundary conditions (4.39), (4.41) (written in terms of $\mathbf{u}^i$).

We shall prove that this system is in the classical framework of semigroup theory. It then suffices to consider the homogeneous system (i.e., with $\mathbf{f} = 0$) corresponding to (4.43), which we write in the form

$$d\mathbf{u}/dt = -\mathcal{A}\mathbf{u}; \qquad \mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3, \mathbf{u}^4). \tag{4.44}$$

We define the space

$$\begin{aligned} \mathscr{V} = \{\mathbf{u}^1, \mathbf{u}^2;\ & \mathbf{u}^1 \in H(\Omega, \operatorname{div}), \mathbf{u}^2 \in \mathbf{L}^2(\Sigma, V), \\ & \mathbf{u}^1 \cdot \mathbf{n}/\partial\Omega = 0,\ (\mathbf{F}, \mathbf{u}^2)_H = u_3^1 \text{ on } \textstyle\sum\}, \end{aligned} \tag{4.45}$$

where we write Ω for Ω^0 and $H(\Omega, \mathrm{div})$ is the classical space formed by the vectors of $\mathbf{L}^2(\Omega)$ whose divergence belongs to $\mathbf{L}^2(\Omega)$ (see, for instance, [12, Chap. 7] for its properties). It is clear that conditions (4.39), (4.41) are included in $\mathscr{V}$; moreover, there are linear relations between traces belonging to $H^{-1/2}(\partial\Omega)$, $L^2(\Sigma)$ and $H^{-1/2}(\Sigma)$ and consequently *$\mathscr{V}$ is a Hilbert space for the norm*

$$\|\mathbf{u}^1, \mathbf{u}^2\|_{\mathscr{V}}^2 = \int_\Omega (|\mathbf{u}^1|^2 + |\mathrm{div}\,\mathbf{u}^1|^2)\, dx + \int_\Sigma \|\mathbf{u}^2\|_V^2\, dx_1\, dx_2. \tag{4.46}$$

We also define

$$\begin{aligned} &\mathscr{H} = \text{adherence of } \mathscr{V} \text{ in } \mathbf{L}^2(\Omega) \times L^2(\Sigma; V), \\ &\|\mathbf{u}^1, \mathbf{u}^2\|_{\mathscr{H}}^2 = \int_\Omega |\mathbf{u}^1|^2\, dx + \int_\Sigma \|\mathbf{u}^2\|_V^2\, dx_1\, dx_2. \end{aligned} \tag{4.47}$$

THEOREM 4.1. *Operator A in* (4.44) *is the generator of a continuous semigroup in $\mathscr{V} \times \mathscr{H}$ (the exact definition of the operator A will becomes clear in the sequel).*

In order to prove this theorem we will apply the classical theorem of Lumer–Phillips. Under the transformation

$$\mathbf{u} = e^{\alpha t}\mathbf{w}$$

the corresponding system for w involves operator $A + \alpha I$ instead of A. Consequently, it suffices to prove that for sufficiently large α, operator $A + \alpha I$, satisfies the following:

(a) its range is the whole space,

(b) it is accretive.

Let us prove conditions (a): Let θ be a given element of $\mathscr{V} \times \mathscr{H}$. We must prove that there exists $\mathbf{u} \in \mathscr{V} \times \mathscr{H}$ satisfying

$$(A + \alpha I)\,\mathbf{u} = \boldsymbol{\theta} \tag{4.48}$$

or, equivalently,:

$$\begin{aligned} \mathbf{u}^3 &= \alpha\mathbf{u}^1 - \boldsymbol{\theta}^1, \\ \mathrm{u}^4 &= \alpha\mathbf{u}^2 - \boldsymbol{\theta}^2 \end{aligned} \tag{4.49}$$

$$\begin{aligned} -\mathbf{grad}\,\mathrm{div}\,\mathbf{u}^1 + \alpha^2\mathbf{u}^1 &= \boldsymbol{\theta}^3 + \alpha\boldsymbol{\theta}^1, \\ [\mathrm{div}\,\mathbf{u}^1]\,\mathbf{F} + \alpha^2\mathbf{u}^2 + \alpha\mathcal{A}\mathbf{u}^2 &= \boldsymbol{\theta}^4 + \alpha\boldsymbol{\theta}^2 + \alpha\mathcal{A}\boldsymbol{\theta}^2. \end{aligned} \tag{4.50}$$

But it is easily seen that (4.50) is equivalent to the following variational problem.

Find $(\mathbf{u}^1, \mathbf{u}^2) \in V$ such that, $\forall (\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2) \in \mathscr{V}$,

$$\int_\Omega \operatorname{div} \mathbf{u}^1 \operatorname{div} \boldsymbol{\varphi}^1 \, dx + \alpha \int_\Sigma ((A + \alpha)\, \mathbf{u}^2, \boldsymbol{\varphi}^2)_H \, dx_1 \, dx_2 + \alpha^2 \int_\Omega \mathbf{u}^1 \cdot \boldsymbol{\varphi}^1 \, dx$$

$$= \int_\Omega (\boldsymbol{\theta}^3 + \alpha \boldsymbol{\theta}^1) \cdot \boldsymbol{\varphi}^1 \, dx + \int_\Sigma (\boldsymbol{\theta}^4, \boldsymbol{\varphi}^2)_H \, dx_1 \, dx_2$$

$$+ \alpha \int_\Sigma ((A + \alpha)\, \boldsymbol{\theta}^2, \boldsymbol{\varphi}^2)_H \, dx_1 \, dx_2. \tag{4.51}$$

It is evident that for large positive α, the left-hand side of (4.51) is a bilinear, continuous and coercive form on $\mathscr{V} \times \mathscr{H}$ and that the right-hand side is linear and continuous on the same space. According to the Lax–Milgram theorem $(\mathbf{u}^1, \mathbf{u}^2) \in \mathscr{V}$ exists and is unique; (4.49) then furnishes $\mathbf{u}^3, \mathbf{u}^4$ and (4.48) has a unique solution.

As for condition (b), we obtain

$$((A + \alpha I)\, \mathbf{u}, \mathbf{u})_{\mathscr{V} \times \mathscr{H}} = -(\mathbf{u}^3, \mathbf{u}^1)_{L^2(\Omega)} - (\mathbf{u}^4, \mathbf{u}^2)_{L^2(\Sigma, V)}$$

$$+ \int_\Sigma (A\mathbf{u}^4, \mathbf{u}^4)_H \, dx_1 \, dx_2 + \alpha \|\mathbf{u}\|^2_{\mathscr{V} \times \mathscr{H}}, \tag{4.52}$$

which is > 0 for sufficiently large α.

5. Acoustic Flow through a Perforated Wall: Two-Dimensional Case

5.1. *Asymptotic Study*

We consider a problem analogous to that of Section 4.1 in the two-dimensional case. The wall is now the axis $x_2 = 0$ (Remark 4.4 still holds). All other notations are the same. Here in Section 3 the dependence in time is taken to be in $e^{-i\omega t}$.

The expansion in the outer region is that of Section 4.2, without modifications.

As for the η-layer, we have again (4.16)–(4.20), but the function φ is not that for a linear distribution of two-dimensional sources (i.e., the classical solution for a source between the two plane walls $x_1 = \pm a/2$):

$$\varphi(\xi) = \frac{D}{2\pi} \log \sin \frac{\pi \xi}{a}, \qquad \xi = y_1 + i y_2. \tag{5.1}$$

The conservation of flux (D for each hole) gives

$$D = \pm \rho a \frac{\partial v_2^{0+}}{\partial t}(x_1, t) \qquad (\pm \text{ for } x_2 \gtrless 0) \tag{5.2}$$

and $\hat{p}^1$ is defined by

$$\begin{aligned} \frac{\partial \hat{p}^1}{\partial y_1} &= \frac{\rho}{2}\frac{\partial v_2^{0+}}{\partial t}(x_1, t)\,\mathrm{Re}\,\mathrm{cotg}\,\frac{\pi\xi}{a} \\ \frac{\partial \hat{p}^1}{\partial y_2} &= -\frac{\rho}{2}\frac{\partial v_2^{0+}}{\partial t}(x_1, t)\,\mathrm{Im}\,\mathrm{cotg}\,\frac{\pi\xi}{a} \end{aligned} \tag{5.3}$$

in the $x_2 > 0$ region (and an analogous expression in $x_2 < 0$).

In order to match (5.3) with the ε-region in the vicinity of the hole, we have

$$\left.\begin{aligned} \frac{\partial \hat{p}^1}{\partial y_1} &\sim \frac{a\rho}{\pi}\frac{\partial v_2^{0+}}{\partial t}(x_1, t)\frac{y_1}{y_1^2 + y_2^2} \\ \frac{\partial \hat{p}^1}{\partial y_2} &\sim \frac{a\rho}{\pi}\frac{\partial v_2^{0+}}{\partial t}(x_1, t)\frac{y_2}{y_1^2 + y_2^2} \end{aligned}\right\} \quad \text{for } |\xi| \to 0 \tag{5.4}$$

and we obtain for $|\xi| \to 0$

$$\hat{p}^1 \sim \pm\rho\frac{a}{\pi}\frac{\partial v_2^{0+}}{\partial t}(x_1, t)\log|y| + c(x, t) \tag{5.5}$$

and consequently, with $(\varepsilon/\eta)z$ instead of y,

$$p^\varepsilon \sim p^{0\pm}(x_1, t) + \eta\left[\pm\frac{\rho a}{\pi}\frac{\partial v_2^{0+}}{\partial t}(x_1, t)\log\frac{\varepsilon}{\eta}|z| + c^1(\mathbf{x}, t)\right], \tag{5.6}$$

where $c^1(x, t)$ denotes an undeterminated function.

On the other hand, in the ε region we have problem (3.2)–(3.6) with $\partial/\partial t$ instead of the factor $-i\omega\rho$ and an undeterminated flux through the hole instead of 1. If χ denotes this fux, Proposition 3.2 gives the following behaviour for large $|z|$.

$$p^\varepsilon \sim \chi\left[\pm\left(\frac{i\omega\rho}{\pi}\log|z| + \Gamma\right) + c^2\right] + \cdots. \tag{5.7}$$

which must match (5.6). The term's log gives

$$\chi = \eta a v_2^0.$$

In order to match the terms of order $O(1)$, we take the difference between the values in the regions $z_2 > 0$ and $z_2 < 0$. We see that *the critical case is*

$$\eta \log \varepsilon = O(1) \Leftrightarrow \varepsilon = ke^{-\beta/\eta}, \tag{5.8}$$

where β and k are positive constant. In this case, the matching gives

$$[p^0] = -2\rho \frac{a\beta}{\pi} \frac{\partial v_2^{0+}}{\partial t} (x_1, t), \tag{5.9}$$

which is the two-dimensional analog of (4.34).

We see that in the two-dimensional case, this law is not of the convolution type. Moreover, it does not depend on the viscosity coefficient λ; λ has only an influence on the constant Γ, which does not appear in (5.9). On the other hand, the constant k in (5.8) does not appear in (5.9); this means that the asymptotic behaviour depends on the order, but not on the effective size of the holes.

Remark 5.1. If $\eta \log \varepsilon$ tends to zero (the holes are "large" with respect to those of the critical case) the appropriate boundary condition is

$$[p^0] = 0$$

and the wall "does not exist" in the limit. Conversely, if $\eta \log \varepsilon$ tends to infinity, the boundary condition is

$$v_2^0 = 0$$

and the wall becomes "impermeable" in the limit.

In the critical case, the outer problem is (4.10)–(4.14) and (5.9). The proof of the existence and uniqueness follows along the same lines as that of Section 4.5, with obvious simplifications.

The equivalent of system (4.43) is here

$$\begin{aligned}
d\mathbf{u}^1/dt &= \mathbf{u}^3, \\
d\mathbf{u}^2/dt &= \mathbf{u}^4, \\
d\mathbf{u}^3/dt &= \mathbf{grad} \operatorname{div} \mathbf{u}^1 + f, \\
d\mathbf{u}^4/dt &= [\operatorname{div} \mathbf{u}^1],
\end{aligned}$$

where unessential constants are taken to be 1. We then prove the existence and uniqueness of the solution in the configuration space analogous to that of Theorem 4.1 but with the spaces V and H replaced by the real line R. In particular $\mathbf{u}^2$ and $\mathbf{u}^4$ are elements of $L^2(\Sigma)$.

5.2. The Case $\varepsilon = \eta$: Flow in the Vicinity of the Wall

In the precesing section (as well as in the three-dimensional case, Section 3) we did not consider the case where ε and η are of the same order. Following Remark 5.1, it is evident that the outer flow is in this case the same as if there is no wall. Nevertheless, the reasoning of the preceding section does not hold, because the two regions (ε and η) coincide.

This case is in the general framework of the homogenization of boundaries (see [12, Chap. 5] for this theory, or Levy and Sanchez [16] for a problem of this kind in acoustics).

We now study this case ($\varepsilon = \eta$) and the structure of the layer in a neighbourhood of Σ. The outer expansion is of course that of Section 4.2. According to the classical theory, we introduce the variable $y = x/\varepsilon$ and we consider

$$\begin{aligned} \mathbf{v}^{\varepsilon} &= \hat{\mathbf{v}}^0(x, y, t) + \varepsilon \hat{\mathbf{v}}^1(x, y, t(+ \cdots, \\ p^{\varepsilon} &= \hat{p}^0(x, y, t) + \varepsilon \hat{p}^1(x, y, t) + \cdots, \end{aligned} \tag{5.11}$$

where the functions are B-periodic with respect to y. The period B is formed by the strip $0 < y_1 < 1$; $-\infty < y_2 < +\infty$ unless a portion of the plane $y_2 = 0$ (i.e., the hollowed wall Σ_0^1). At the first order we obtain

$$\operatorname{div}_y \mathbf{v}^0 = 0, \tag{5.12}$$

$$\mathbf{grad}_y\, \hat{p}^0 = 0, \tag{5.13}$$

$$\rho \frac{\partial \hat{\mathbf{v}}^0}{\partial t} = -\mathbf{grad}_x\, \hat{p}^0 - \mathbf{grad}_y\, \hat{p}^1 + \lambda\, \Delta_y \hat{\mathbf{v}}^0. \tag{5.14}$$

$\hat{\mathbf{v}}^0$ must match with $\mathbf{v}^0(x)$ for $y_2 \to \pm\infty$. Equation (5.12) expresses the fact that the flux of $\hat{v}^0$ through any closed surface is zero. Then, taking into account the periodicity conditions, we have

$$[v_2^0] = 0. \tag{5.15}$$

The matching of $\hat{p}^0$ gives

$$\hat{p}^0(x, t) = p^{0\pm}(x, t). \tag{5.16}$$

Consequently

$$[p^0] = 0. \tag{5.17}$$

This, paired wth (5.15) shows that *the outer flow is the same as if the wall does not exist.* We go further to the study of the layer.

By differentiating (5.17) and using (5.14) at infinity, we have

$$\left[\frac{\partial \hat{p}^0}{\partial x_1}\right] = 0 \Rightarrow [\mathbf{v}^0] = 0. \tag{5.18}$$

In order to write (5.12)–(5.14) as a problem with homogeneous boundary conditions at infinity, we define an auxiliary vector $\mathbf{V}(x, y, t)$ satisfying

$$\begin{gathered} \operatorname*{div}_y \mathbf{V} = 0, \\ \mathbf{V}|_{y_2 = \pm\infty} = \mathbf{v}^0(x, t), \\ \mathbf{V}|_{\Sigma_0} = 0, \\ \mathbf{V} \text{ is } B\text{-periodic,} \end{gathered} \tag{5.19}$$

which exists because the compatibility condition (5.15) is satisfied. We consider the new unknown

$$\mathbf{w}^0(x, y, t) = \hat{\mathbf{v}}^0(x, y, t) - \mathbf{V}(x, y, t) \tag{5.20}$$

and the problem becomes

$$\operatorname*{div}_y \mathbf{w}^0 = 0, \tag{5.21}$$

$$\rho \frac{\partial \mathbf{w}^0}{\partial t} = -\mathbf{grad}_y\, \hat{p}^1 + \lambda \Delta_y \mathbf{w}^0 + \mathbf{F}, \tag{5.22}$$

$$\mathbf{w}^0|_\infty = 0, \qquad \mathbf{w}^0|_\Sigma = 0. \tag{5.23}$$

$\hat{p}^1$, $\mathbf{w}^0$, are B-periodic, where $\mathbf{F}(x, y, t)$ is a given function taking the value zero for large $|y|$. In order to study this problem we define the functional spaces

$$\mathscr{V} = \{\mathbf{u}; \mathbf{u}|_\infty = 0, \mathbf{u}|_{\Sigma_0} = 0, \operatorname{div} \mathbf{u} = 0, \mathbf{u}\ \ B\text{-periodic}\}.$$

V is the completion of $\mathscr{V}$ for the norm of $\mathbf{H}^1(B)$. H is the completion of $\mathscr{V}$ for the norm of $\mathbf{L}^2(B)$.

Then, (5.22) is equivalent to find a function $\mathbf{w}^0$ of t with values in V such that

$$\rho \left(\frac{d\mathbf{w}^0}{dt}, u\right)_H = -\lambda(\mathbf{w}^0, \mathbf{u})_V + \lambda(\mathbf{w}^0, \mathbf{u})_H + (\mathbf{F}, \mathbf{u})_H \qquad \forall \mathbf{u} \in V.$$

This problem is in the standard framework of semigroup theory for parabolic equations. $\mathbf{w}^0$ exists and is unique when the initial value (see (5.23)) is given.

Note added in proof. The proof of Lemma 2.3 is more clear if we consider the regions $z_3 > Q$, $z_3 < 0$ (instead of $z_3 > 1$, $z_3 < -1$). Moreover, from the given references, in particular [1] p. 74 we know that *grad* Q may be approximated in each region by a function which is zero for sufficiently large $|z|$, and this easily furnishes the last equality of the proof.

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